International Journal of Theoretical Physics, Vol. 16, No. 10 (1977), pp. 755–761

Spin in Kink-Type Field Theories

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Received: 19 August 1977

Abstract

This paper studies some classical three-dimensional field theories for which the ranges of the field variables are a 3-sphere, a 2-sphere, the symplectic group, $Sp(n)$, the special orthogonal group, $SO(3)$, and the $S_{4,1}$ space of general relativistic metrics. The main result is the proof that these theories admit half-odd-integer spin, so that the 1-kink states are classical analogs of fermion states.

1. *Introduction*

For the purposes of this paper, a three-dimensional classical field theory is specified by giving the manifold Y into which R^3 is mapped by the field variables. Thus any such theory involves mappings φ,

$$\varphi : R^3 \to Y$$

Furthermore, we shall only consider mappings φ and homotopies φ_t of φ for which

$$\varphi(\mathbf{x}) \to y_0, \qquad \varphi_t(\mathbf{x}) \to y_0 \qquad \text{as } |\mathbf{x}| \to \infty$$

where $\mathbf{x} \in R^3$ and y_0 is some fixed base point of Y. Such mappings are of course equivalent to base-point-preserving mappings of S^3, the one-point compactification of R^3, into Y. By referring to a "Y theory" we shall mean a field theory of the above type. The most interesting situations arise for Y theories for which Y is topologically nontrivial, and, in particular, in which the third homotopy group of Y is not isomorphic to the trivial group of one element. We follow the terminology of Finkelstein (1966) in making the following definition.

Definition. A Y theory for which $\pi_3(Y) \neq 0$ is said to "admit kinks."

Theories that admit kinks were first studied by Enz (1963), Finkelstein and Misner (1959), and Skryme (1958). Given that a Y theory admits kinks, the space of all mappings φ is the disjoint union of a set of equivalence classes such that two mappings belong to the same equivalence class if and only if they are homotopic. These equivalence classes are the group elements of $\pi_3(Y)$. One or more of the equivalence classes will be generators for $\pi_3(Y)$. Mappings belonging to such equivalence classes are called 1-kink mappings. Similarly, mappings belonging to an equivalence class resulting from applying a generator n times to the identity element of $\pi_3(Y)$ are called n-kink mappings.

For three-dimensional theories, it is sensible to talk about spin. One of the most interesting possibilities arising with three-dimensional theories that admit kinks is that the 1-kink mappings in the classical theory might correspond to half-odd-integer spin states when the theory is quantized. If this is so, then a kink can be regarded as a classical analog of a fermion. This idea is made more precise in the following definition (Finkelstein and Rubinstein, 1968).

> *Definition.* If Q_1 denotes an equivalence class(es) of 1-kink mappings for a Y theory, then the Y theory is said to "admit half-odd-integer spin" provided that the following conditions are satisfied:
> (i) $\pi_1(Q_1)$ has an element of order 2.
> (ii) The 2π rotation paths in Q_1 are nontrivial.

Condition (i) is redundant, since it is implied by condition (ii). However, condition (i) can be used as a preliminary test for the possibility of admission of half-odd-integer spin since it is usually easy to check, as can be seen from the following argument.

Let Q_0 denote the identity element of $\pi_3(Y)$. G. W. Whitehead (1946) has established the following isomorphism:

$$\pi_1(Q_1) \approx \pi_1(Q_0)$$

It is also well known (Hilton and Wylie, 1967, proposition II.2.5) that

$$\pi_1(Q_0) \approx \pi_4(Y)$$

Hence condition (i) is readily checked by evaluating the fourth homotopy group of Y.

The purpose of this paper is to study the special cases for which Y is any of $S^3, S^2, Sp(n), SO(3)$, or $S_{4,1}$. $S_{4,1}$ denotes the set of 4×4 real symmetric matrices that are similar to diag $(1, 1, 1, -1)$, and is relevant to the space of metrics in general relativity. For each of these cases,

$$\pi_3(Y) \approx Z$$

where Z denotes the group of integers. Thus each of these theories admits kinks. Also, for each of the above cases,

$$\pi_4(Y) \approx Z_2$$

where Z_2 is the group of integers modulo 2. Hence it is reasonable to investigate the possibility of half-odd-integer spin. The main result of this paper is the proof that each of these theories admits half-odd-integer spin. This result has already been obtained for S^3 and $S_{4,1}$ (Williams, 1970, 1971), but the method of proof presented in this paper is new and more systematic than previous approaches.

2. *The S^3 Theory*

The 3-sphere can be parametrized by four real variables $(\phi_1, \phi_2, \phi_3, \phi_4)$ subject to the restriction

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = 1$$

Let $(0, 0, 0, 1)$ be the fixed point of S^3. Hence the theory involves mappings φ,

$$\varphi : R^3 \to S^3$$

with

$$\varphi(\mathbf{x}) \to (0, 0, 0, 1) \qquad \text{as } |\mathbf{x}| \to \infty$$

The "kink number" is simply the degree of the mapping. An example of a 1-kink mapping is given by the following mapping s:

$$s : R^3 \to S^3$$

with

$$s(x) = (\phi_1, \phi_2, \phi_3, \phi_4)$$

and

$$\phi_i = 2ax_i/(r^2 + a^2), \qquad i = 1, 2, 3$$
$$\phi_4 = (r^2 - a^2)/(r^2 + a^2)$$

where $r = \|\mathbf{x}\|$ and a is a constant parameter. This mapping is the familiar stereographic projection. For kink field theories, it is usual to consider the center of the kink to be at the point(s) of R^3 that maps into that point of Y that has the largest metric distance from the fixed point. For the above example, $\mathbf{x} = \mathbf{0}$ is the point that is mapped into $(0, 0, 0, -1)$, and hence the kink is centered at the origin of R^3. An example of a theory of the above type has been discussed at length by Skyrme (1958, 1961, 1962, 1971).

3. *The S^2 Theory*

The 2-sphere can be parametrized by three real variables (ψ_1, ψ_2, ψ_3) with

$$\psi_1^2 + \psi_2^2 + \psi_3^2 = 1$$

To understand the nature of the kinks of the S^2 theory, it is convenient to consider the topologically equivalent problem of mapping S^3 into S^2. The invariant characterizing the mappings is called the Hopf invariant. Define h, the Hopf mapping (Hilton, 1966), as follows:

$$h : S^3 \to S^2$$

with

$$h(\phi_1, \phi_2, \phi_3, \phi_4) = (\psi_1, \psi_2, \psi_3)$$

where

$$\begin{aligned}\psi_1 &= 2(\phi_3\phi_1 - \phi_4\phi_2)\\ \psi_2 &= 2(\phi_3\phi_2 + \phi_4\phi_1)\\ \psi_3 &= 1 - 2(\phi_1^2 + \phi_2^2)\end{aligned}$$

An example of a 1-kink mapping for the S^2 theory is then provided by hs, the composition of the stereographic projection with the Hopf mapping. One of the most interesting features of the S^2 theory is that the high-energy density region of the kink has the form of a loop in R^3. This loop is the set $\{\mathbf{x} \mid \mathbf{x} \to (0, 0, -1)\}$, and for the mapping hs it is simply the circle $x_1^2 + x_2^2 = a^2, x_3 = 0$. Enz (1977) has recently studied extended particles for a field theory whose range is the 2-sphere. He has studied the topology in some detail, although his boundary conditions are different from ours, leading to a situation in which the mappings are characterized by more than one type of invariant.

4. *The Sp(n) and SO(3) Theories*

The symplectic group $Sp(n)$ is defined as the group of linear transformations in quaternionic n-dimensional space H^n that preserve the inner product. Since H^n is equivalent to complex $2n$-dimensional space, $Sp(n)$ can also be defined as the group of $2n \times 2n$ complex unitary matrices U such that

$$U^t \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} U = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

where 1_n denotes the $n \times n$ unit matrix. We shall follow the terminology of Steenrod (1951) and Husemoller (1974) [although some authors (see, for example, Weyl, 1939) refer to the above group as the *unitary* symplectic group, and denote it by $USp(2n)$].

$Sp(1)$ is homeomorphic to S^3. For $n > 1$, $Sp(n)$ contains $Sp(n-1)$ as a subgroup, and so by induction contains $Sp(1)$ as a subgroup. It follows (Steenrod, 1951, p. 132) that any degree-1 mapping from R^3 onto the $Sp(1)$ subspace of $Sp(n)$ is an example of a 1-kink mapping for the $Sp(n)$ theory.

For $SO(3)$, the special orthogonal group in three-dimensions, the double covering $S^3 \to SO(3)$ provides an example of a 1-kink mapping.

5. *Basic Facts about the J Homomorphism*

The homomorphism $J: \pi_i(SO(n)) \to \pi_{n+i}(S^n)$ was introduced by G. W. Whitehead (1942) and has been of great importance in homotopy theory. We give a brief description of this homomorphism and its relevant properties.

For any $\xi = (\phi_1, \ldots, \phi_{i+n+1}) \in S^{i+n}$, write $\xi = (x, y)$, where $x = (\phi_1, \ldots, \phi_{i+1})$, $y = (\phi_{i+2}, \ldots, \phi_{i+n+1})$, and also $\|x\| = \cos\frac{1}{2}\pi t$, $\|y\| = \sin\frac{1}{2}\pi t$, $0 \leqslant t \leqslant 1$ (which determines t uniquely). Regard S^n as the suspension ΣS^{n-1}, so that points of S^n are of the form $[z, t]$, $z \in S^{n-1}$, $0 \leqslant t \leqslant 1$, with $[z, 0] = [z', 0]$ and $[z, 1] = [z', 1]$ $\forall z, z' \in S^{n-1}$.

Now let $f: S^i \to SO(n)$, and define

$$\tilde{f}: S^i \times S^{n-1} \to S^{n-1}$$

by

$$\tilde{f}(u, v) = f(u)(v)$$

There is a well-defined map

$$g: S^{i+n} \to S^n$$

given by

$$g(\xi) = [\tilde{f}(x/\|x\|, y/\|y\|), t]$$

It is fairly clear that the homotopy class of g depends only on the homotopy class of f, and one sets $[g] = J[f]$.

Proofs that J is in fact a homomorphism as well as other basic facts about J can be found in Whitehead's paper (1942) or in Husemoller's book (1974). Whitehead proves that J gives an isomorphism on the 1-stem, i.e.,

$$J: \pi_1(SO(n)) \xrightarrow{\approx} \pi_{n+1}(S^n)$$

Note that the group in question here equals Z if $n = 2$, and Z_2 for $n \geqslant 3$. We shall also make use of the following commutative diagram (Whitehead, 1942):

$$\begin{array}{ccc} \pi_i(SO(n)) & \xrightarrow{J} & \pi_{n+i}(S^n) \\ \downarrow{\scriptstyle \mu_{n*}} & & \downarrow{\scriptstyle E} \\ \pi_i(SO(n+1)) & \xrightarrow{J} & \pi_{n+i+1}(S^{n+1}) \end{array}$$

where $\mu_n: SO(n) \hookrightarrow SO(n+1)$ and E is the Freudenthal suspension homomorphism (Steenrod, 1951).

6. *2π Rotation Paths*

For $n > 1$, let $\varphi: S^n \to Y$ represent a generator of $\pi_n(Y)$; i.e., φ is a 1-kink map. The base point of S^n is $s_0 = (0, 0, \ldots, 1)$, that of Y is denoted by y_0, and, following Section 1, we write Q_1 for the space of all base-point-preserving maps that are homotopic to φ.

Definition. A 2π rotation path in Y (of dimension n) is the loop $\omega : S^1 \to Q_1$ given by

$$\omega(t)(\phi_1, \ldots, \phi_{n+1}) = \varphi((\phi_1, \ldots, \phi_{n+1}) \cdot R_t)$$

$0 \leqslant t \leqslant 1$, where

$$R_t = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t & \\ & & 0 \\ -\sin 2\pi t & \cos 2\pi t & \\ & 0 & I \end{pmatrix}$$

Notice $\omega(t)(s_0) = \varphi(s_0 \cdot R_t) = \varphi(s_0) = y_0 \ \forall t$, and also $\omega(0) = \omega(1) = \varphi$.

Let $\iota_1 : S^1 \xrightarrow{\approx} SO(2)$ be the map identifying S^1 with $SO(2)$. As remarked in Section 1, $\pi_1(Q_1) \approx \pi_1(Q_0) \approx \pi_{n+1}(Y)$, and we are interested, in the case $n = 3$, in establishing the nontriviality of $[\omega] \in \pi_1(Q_1)$. First consider the case $Y = S^3$. Then we take φ to be the identity map $S^3 \to S^3$, and note $\omega = \epsilon\mu_3\mu_2\iota_1 : S^1 \to Q_1$, where ι_1, μ_2, μ_3 are as above and $\epsilon : SO(4) \hookrightarrow Q_1$ is the inclusion.

Proposition. $[\omega]$ corresponds to the nonzero element of $\pi_4(S^3)$.

Proof. From the result of Whitehead's quoted in Section 5 above it follows that $J[\mu_3\mu_2\iota_1] \neq 0$ in $\pi_5(S^4) \approx Z_2$. The result to be proved then follows immediately on applying the following factorization of J to $[\mu_3\mu_2\iota_1]$. (Husemoller, 1974, p. 212):

$$\pi_1(SO(4)) \xrightarrow{\epsilon*} \pi_1(Q_1) \xrightarrow[\approx]{\theta} \pi_4(S^3) \xrightarrow[\approx]{E} \pi_5(S^4)$$

since $[\omega] = \epsilon_*[\mu_3\mu_2\iota_1]$.

Corollary 1. The S^3 theory admits half-odd-integer spin.

Corollary 2. The S^2 theory admits half-odd-integer spin.

Proof of Corollary 2. Simply consider the diagram

$$\pi_1(SO(4)) \xrightarrow{\epsilon_*} \pi_1(Q_1) \xrightarrow[\approx]{\theta} \pi_4(S^3) \xrightarrow[\approx]{h_*} \pi_4(S^2)$$

Since $[\omega] \in \pi_1(Q_1)$ is nonzero and h_*, θ are isomorphisms, the relevant class $h_*\theta[\omega]$ is nonzero here.

Corollary 3. The $Sp(n)$-theory admits half-odd-integer spin, $n \geqslant 1$.

Proof of Corollary 3. This is proved in the same manner as Corollary 2, by composing $\theta\epsilon_*$ with the isomorphisms (Steenrod, 1951, p. 132):

$$\pi_4(S^3) \approx \pi_4(Sp(1)) \xrightarrow{\approx} \pi_4(Sp(2)) \xrightarrow{\approx} \cdots \xrightarrow{\approx} \pi_4(Sp(n))$$

Corollary 4. The $SO(3)$ theory admits half-odd-integer spin.

Proof of Corollary 4. Again this follows since the double covering $c : S^3 \to SO(3)$ induces an isomorphism $c_* : \pi_4(S^3) \xrightarrow{\approx} \pi_4(SO(3))$. This result is also given by Finkelstein (1966).

The $SO(3)$ theory is closely related to the particular general relativistic case in which the metric defines a base-point-preserving mapping from S^3 to $S_{4,1}$. (In specifying the space-time manifold to be S^3, we are of course being very restrictive.) The existence of a fibration $S_{4,1} \to SO(3)$ with contractible fiber (Steenrod, 1951, Chap. 40) shows that the $S_{4,1}$ theory is homotopically equivalent to the $SO(3)$ theory. Thus the $S_{4,1}$ theory admits kinks and half-odd-integer spin.

7. *Summary and Conclusions*

This paper has studied a number of nonlinear field theories. We have investigated the topological structure of these theories, without assuming any particular forms of Lagrangian, and have shown that the theories admit half-odd-integer spin. Thus the 1-kink states should correspond to fermion states in the corresponding quantized theories.

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